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Soft Directed Graphs: Exploring Categorical Product

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Abstract

Molodtsov pioneered the soft set theory, offering a mathematical framework tailored for managing uncertain data, a concept now widely embraced by scholars to address decision-making challenges. Directed graphs, comprising nodes connected by directed edges, serve as invaluable tools for analysing and resolving problems related to social connections, optimal routes, electrical circuits, and various other domains. Extending the notion of soft sets to directed graphs, soft directed graphs provide a parameterised approach for understanding relationships within directed graph structures. This study delves into the exploration and analysis of certain characteristics of the categorical and restricted categorical products of soft directed graphs. We establish that these products form soft directed graphs. Detailed results are provided on the node and directed edge counts of both products, along with insights into the directed part in-degree, out-degree, and degree sums of their nodes, enhancing the theoretical foundation of soft directed graphs.

Keywords: soft graph; soft directed graph; categorical product.

1 Introduction

Graphs are fundamental structures used to model relationships between entities in various real-world scenarios. Their versatility lies in their ability to abstract complex connections into simple representations. In many applications, graphs are indispensable, offering solutions to problems in fields ranging from social networking to transportation logistics and beyond.

A graph consists of two main components: nodes and edges. Directed Graphs, a specific type of graph, have edges with a directionality, meaning they indicate a one-way relationship between nodes. In practical applications, graphs are encountered in diverse contexts. Social media platforms utilise graphs to represent connections between users, enabling functionalities like friend recommendations and social network analysis. Navigation systems like Google Maps utilise graphs to represent road networks, allowing users to find optimal routes between locations. The internet itself can be modelled as a graph, with web pages as nodes and hyperlinks as edges. Blockchains, the underlying technology behind cryptocurrencies, also use graphs to represent transaction histories and verify transactions. Additionally, graphs play a crucial role in neural networks, where they represent the connections between artificial neurons.

The concept of soft sets, introduced by Molodtsov [14], extends the traditional set theory to handle uncertainties. Soft set theory provides a mathematical framework for dealing with imprecise or uncertain information, making it useful in solving problems where traditional mathematical tools fall short. Researchers like Adam and Hassan [1] and Kamal and Abdulla [13] have successfully applied this theory to various practical problems. Maji et al. [15] also explained a fuzzy soft set theoretic approach to decision-making problems.

Building on the foundation of soft set theory, researchers such as Thumbakara and George [19] introduced the concept of soft graphs. They also introduced subdivision graph, power, and line graph of a soft graph. Soft graphs extend the traditional graph model to incorporate uncertainty, enabling the representation and analysis of uncertain relationships between entities. Akram and Nawas [4] modified the definition of soft graphs and introduced certain types of soft graphs [5], fuzzy soft graph [3], and some of their applications [6]. Akram and Zafar studied soft trees [7] and fuzzy soft trees [2]. Advancements in the field of soft graphs have been significant. Researchers Thenge, Jain, and Reddy have contributed to the development of soft graphs, by introducing concepts like connectedness [18], soft trees [17], and associated matrices [16]. George, Thumbakara, and Jose have further expanded the domain by introducing concepts such as soft hypergraphs, soft directed graphs [12], and soft disemigraphs, and thoroughly investigating their properties and applications.

The study of soft graphs has also led to the exploration of graph product operations [10]. Product operations allow the combination of two graphs to create a new graph with specific properties. Additionally, Baghernejad and Borzooei [8] have demonstrated the utility of soft graphs and soft multigraphs in managing complex systems such as urban traffic flows. Further contributions to the field include the introduction of novel concepts such as Eulerian and Hamiltonian soft graphs, graph isomorphism, and various product operations on soft graphs like tensor and strong products and co-normal and modular products. They also introduced some topics in soft directed graphs like associated degrees and matrices [11], modular product, homomorphic product, rooted product, disjunctive product and corona product. Additionally, they have extended these concepts to semigraphs and introduced soft semigraphs, and studied their connectedness, and some operations.

The study of soft graphs represents a significant advancement in graph theory, enabling the representation and analysis of uncertain relationships in complex systems. The application of soft set theory to graphs opens up new possibilities for solving practical problems in diverse fields. In this work, the categorical and the restricted categorical products of soft directed graphs are introduced and studied.

2 Preliminaries

In this preliminary section, we lay the foundation for comprehending soft sets, directed graphs, and soft directed graphs. Also, we provide a brief overview of topics including the directed part and various types of degrees associated with soft directed graphs.

2.1 Directed graphs

For preliminaries of directed graphs, we refer to [9].

"A *directed graph* or *directed graph* Ψ^* consists of a non-empty finite set ϱ of elements called *nodes* and a finite set δ of ordered pairs of distinct nodes called *directed edges* or *arcs*. We often write $\Psi^* = (\varrho, \delta)$ to represent a directed graph. The number of nodes and directed edges in a directed graph Ψ^* are called *order* and *size* respectively. The first node u of a directed edge (u, v) is called its *tail* and the second node v is called its *head*. If (u, v) is a directed edge then v is *adjacent from* u and u is *adjacent to* v. A node u is *incident* to a directed edge a if u is the head or tail of a.

A directed graph $\Psi^{**} = (\varrho', \delta')$ is called a *subdirected graph* of $\Psi^* = (\varrho, \delta)$ if $\varrho' \subseteq \varrho$ and $\delta' \subseteq \delta$. The *in-degree* of a node v denoted by $ideg\ v$ is the number of nodes in Ψ^* from which v is adjacent and *out-degree* of v denoted by $odeg\ v$ is the number of nodes in Ψ^* to which v is adjacent. The sum $ideg\ v + odeg\ v$ is called the *degree* of the node v and is denoted by $deg\ v$. In a directed graph $\Psi^* = (\varrho, \delta)$, $\sum_{v\in\varrho} ideg(v) = \sum_{v\in\varrho} odeg(v) =$ Number of directed edges in Ψ^* and $\sum_{v\in\varrho} deg(v) = 2$ (Number of directed edges in Ψ^*). Let $\Psi_1^* = (\varrho_1, \delta_1)$ and $\Psi_2^* = (\varrho_2, \delta_2)$ be two directed graphs. Their *categorical product* $\Psi_1^* \times \Psi_2^*$ is a directed graph with node set $\varrho(\Psi_1^* \times \Psi_2^*) = \varrho_1 \times \varrho_2$ and directed edge set $\delta(\Psi_1^* \times \Psi_2^*)$, where $((t_1, t_1'), (t_2, t_2'))$ is a directed edge in $\Psi_1^* \times \Psi_2^*$ if and only if (t_1, t_2) is a directed edge in Ψ_1^* and (t_1', t_2') is a directed edge in Ψ_2^*. "

2.2 Soft set

Molodstov [14] defined soft set as follows:

"Let R be a set of parameters and U be an initial universe set. Then a pair (F, R) is called a *soft set* (over U) if and only F is a mapping of R into the power set of U. That is, $F : R \to \mathcal{P}(U)$."

2.3 Soft directed graphs

Jose et al. [11] defined soft directed graph as follows:

"Let $\Psi^* = (\varrho, \delta)$ be a directed graph having node set ϱ and directed edge set δ and let $\Re$ be a non-empty set. Let a subset R of $\Re \times \varrho$ be an arbitrary relation from $\Re$ to ϱ. Define $\gamma : \Re \to \mathcal{P}(\varrho)$ by $\gamma(\varepsilon) = \{u \in \varrho | \varepsilon R u\}$ where $\mathcal{P}(\varrho)$ denotes the powerset of ϱ. The pair $(\gamma, \Re)$ is a soft set over ϱ. Also, define $\alpha : \Re \to \mathcal{P}(\delta)$ by $\alpha(\varepsilon) = \{(u, v) \in \delta | \{u, v\} \subseteq \gamma(\varepsilon)\}$ where $\mathcal{P}(\delta)$ denotes the powerset of δ. The pair $(\alpha, \Re)$ is a soft set over the directed edge set δ.

Then $\Psi = (\Psi^*, \gamma, \alpha, \Re)$ is called a soft directed graph if it satisfies the following conditions:

1. $\Psi^* = (\varrho, \delta)$ is a directed graph having node set ϱ and directed edge set δ.
2. $\Re$ is a nonempty set of parameters.
3. $(\gamma, \Re)$ is a soft set over the node set ϱ.
4. $(\alpha, \Re)$ is a soft set over the directed edge set δ.
5. $(\gamma(\varepsilon), \alpha(\varepsilon))$ is a subdirected graph of Ψ^* for all $\varepsilon \in \Re$.

If we represent $(\gamma(\varepsilon), \alpha(\varepsilon))$ by $M(\varepsilon)$ then the soft directed graph Ψ is also given by $\{M(\varepsilon) : \varepsilon \in \Re\}$. Then $M(\varepsilon)$ corresponding to a parameter ε in $\Re$ is called a *directed part* or simply *dipart* of the soft directed graph Ψ.

Let $\Psi = (\Psi^*, \gamma, \alpha, \Re)$ be a soft directed graph and let $M(\varepsilon)$ be a directed part of Ψ for some $\varepsilon \in \Re$. Let v be a node of $M(\varepsilon)$. Then directed part indegree of v in $M(\varepsilon)$ denoted by $ideg\ v[M(\varepsilon)]$ is defined as the number of nodes of $M(\varepsilon)$ from which v is adjacent. That is, $ideg\ v[M(\varepsilon)]$ is the number of directed edges of $M(\varepsilon)$ that have v as its head. Similarly, directed part outdegree of v in $M(\varepsilon)$ denoted by $odeg\ v[M(\varepsilon)]$ is defined as the number of nodes of $M(\varepsilon)$ to which v is adjacent. That is, $odeg\ v[M(\varepsilon)]$ is the number of directed edges of $M(\varepsilon)$ that have v as its tail. The directed part degree of v in $M(\varepsilon)$ is defined as the sum, $ideg\ v[M(\varepsilon)] + odeg\ v[M(\varepsilon)]$ and is denoted by $deg\ v[M(\varepsilon)]$."

3 Categorical Product of Soft Directed Graphs

In this section, we explore the concept of the categorical product of soft directed graphs. Starting with two directed graphs, we derive their corresponding soft directed graphs and define their categorical product. We establish three key theorems: one demonstrating that the categorical product of soft directed graphs is itself a soft directed graph; another detailing the node and directed edge count in the resultant graph; and the third theorem offers insights into various degree sums related to the categorical product. Examples are provided to illustrate these concepts.

Definition 3.1. *Let,* $\Psi_1^* = (\varrho_1, \delta_1)$ *and* $\Psi_2^* = (\varrho_2, \delta_2)$ *be two directed graphs and*

$$\Psi_1 = (\Psi_1^*, \gamma_1, \alpha_1, \Re_1) = \{M_1(\varepsilon) : \varepsilon \in \Re_1\}, \quad \text{and} \quad \Psi_2 = (\Psi_2^*, \gamma_2, \alpha_2, \Re_2) = \{M_2(\varepsilon) : \varepsilon \in \Re_2\},$$

be two soft directed graphs of Ψ_1^* *and* Ψ_2^* *respectively. Then the categorical product of* Ψ_1 *and* Ψ_2*, which is represented by* $\Psi_1 \times \Psi_2$*, is defined as* $\Psi_1 \times \Psi_2 = \{M_1(\varepsilon_1) \times M_2(\varepsilon_2) : (\varepsilon_1, \varepsilon_2) \in \Re_1 \times \Re_2\}$*. Here* $M_1(\varepsilon_1) \times M_2(\varepsilon_2)$ *denotes the categorical product of the directed parts* $M_1(\varepsilon_1)$ *of* Ψ_1 *and* $M_2(\varepsilon_2)$ *of* Ψ_2 *which is defined as follows:*

$M_1(\varepsilon_1) \times M_2(\varepsilon_2)$ *is a directed graph with node set* $\varrho(M_1(\varepsilon_1) \times M_2(\varepsilon_2)) = \gamma_1(\varepsilon_1) \times \gamma_2(\varepsilon_2)$ *and directed edge set* $\delta(M_1(\varepsilon_1) \times M_2(\varepsilon_2))$*, where* $((t_1, t_1'), (t_2, t_2'))$ *is a directed edge in* $M_1(\varepsilon_1) \times M_2(\varepsilon_2)$ *if and only if* (v_1, v_2) *is a directed edge in* $M_1(\varepsilon_1)$ *and* (t_1', t_2') *is a directed edge in in* $M_2(\varepsilon_2)$*.*

Example 3.1. *Let $\Psi_1^* = (\varrho_1, \delta_1)$ be a directed graph which is shown in Figure 1.*

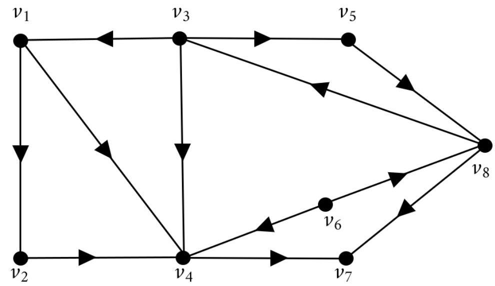

Figure 1: Directed graph $\Psi_1^* = (\varrho_1, \delta_1)$.

Let $\Re_1 = \{v_1, v_6\} \subseteq \varrho_1$ be a set of parameters. Define $\gamma_1 : \Re_1 \to \mathcal{P}(\varrho_1)$ by

$$\gamma_1(\varepsilon) = \{u \in \varrho_1 \mid u = \varepsilon \text{ or } u \text{ is adjacent from } \varepsilon\}, \qquad \forall \varepsilon \in \Re_1.$$

That is, $\gamma_1(v_1) = \{v_1, v_2, v_4\}$ and $\gamma_1(v_6) = \{v_4, v_6, v_8\}$. Here $(\gamma_1, \Re_1)$ is a soft set over ϱ_1.

Also, define $\alpha_1 : \Re_1 \to \mathcal{P}(\delta_1)$ by $\alpha_1(\varepsilon) = \{(u, v) \in \delta_1 \mid \{u, v\} \subseteq \gamma_1(\varepsilon)\}, \forall \varepsilon \in \Re_1$. That is, $\alpha_1(v_1) = \{(v_1, v_2), (v_1, v_4), (v_2, v_4)\}$ and $\alpha_1(v_6) = \{(v_6, v_4), (v_6, v_8)\}$. Here, $(\alpha_1, \Re_1)$ is a soft set over δ_1. Then $M_1(v_1) = (\gamma_1(v_1), \alpha_1(v_1))$ and $M_1(v_6) = (\gamma_1(v_6), \alpha_1(v_6))$ are subdirected graphs of Ψ_1^* as shown in Figure 2. Therefore $\Psi_1 = \{M_1(v_1), M_1(v_6)\}$ is a soft directed graph of Ψ_1^*.

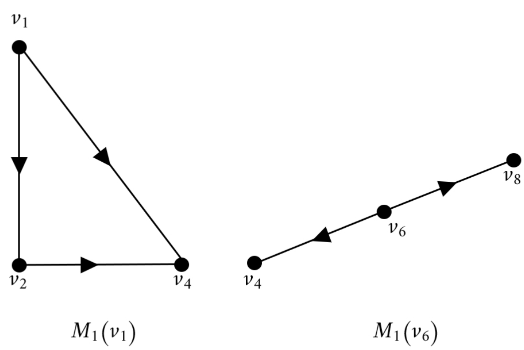

Figure 2: Soft directed graph $\Psi_1 = \{M_1(v_1), M_1(v_6)\}$.

Let $\Psi_2^* = (\varrho_2, \delta_2)$ be a directed graph which is shown in Figure 3.

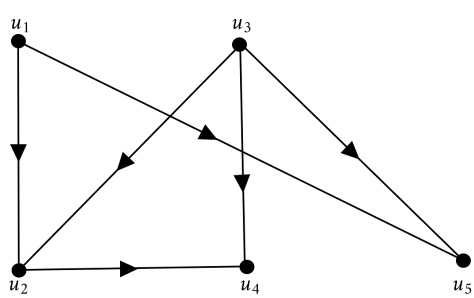

Figure 3: Directed graph $\Psi_2^* = (\varrho_2, \delta_2)$.

Consider the parameter set $\Re_2 = \{u_2\} \subseteq \varrho_2$. Define $\gamma_2 : \Re_2 \to \mathcal{P}(\varrho_2)$ by

$$\gamma_2(\varepsilon) = \{u \in \varrho_2 \mid u = \varepsilon \text{ or } u \text{ is adjacent from } \varepsilon\}, \qquad \forall \varepsilon \in \Re_2.$$

That is, $\gamma_2(u_2) = \{u_2, u_4\}$. Here, $(\gamma_2, \Re_2)$ is a soft set over ϱ_2. Also, define $\alpha_2 : \Re_2 \to \mathcal{P}(\delta_2)$ by $\alpha_2(\varepsilon) = \{(u, v) \in \delta_2 \mid \{u, v\} \subseteq \gamma_2(\varepsilon)\}, \forall \varepsilon \in \Re_2$. That is, $\alpha_2(u_2) = \{(u_2, u_4)\}$. Here, $(\alpha_2, \Re_2)$ is a soft set over δ_2. Then, $M_2(u_2) = (\gamma_2(u_2), \alpha_2(u_2))$ is a subdirected graph of Ψ_2^* as shown in Figure 4. Therefore, $\Psi_2 = \{M_2(u_2)\}$ is a soft directed graph of Ψ_2^*.

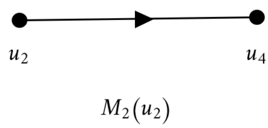

Figure 4: Soft directed graph $\Psi_2 = \{M_2(u_2)\}$.

Then, $\Psi_1 \times \Psi_2 = \{M_1(v_1) \times M_2(u_2), M_1(v_6) \times M_2(u_2)\}$ is shown in Figure 5.

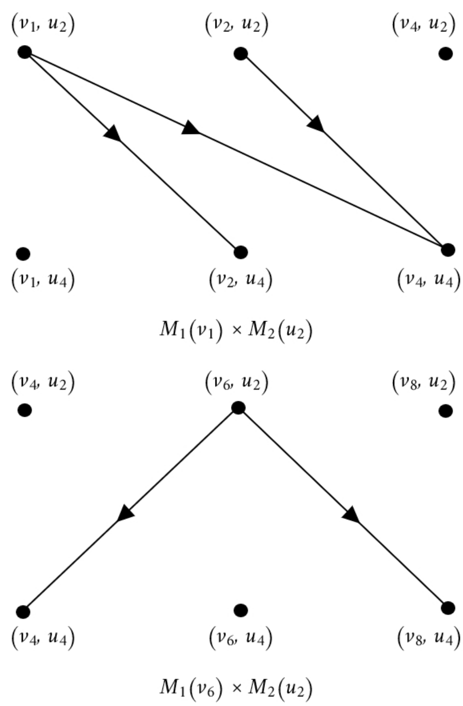

Figure 5: $\Psi_1 \times \Psi_2 = \{M_1(v_1) \times M_2(u_2), M_1(v_6) \times M_2(u_2)\}$.

Theorem 3.1. *Let $\Psi_1^* = (\varrho_1, \delta_1)$ and $\Psi_2^* = (\varrho_2, \delta_2)$ be two directed graphs and Ψ_1 and Ψ_2 be two soft directed graphs of Ψ_1^* and Ψ_2^* respectively. Then $\Psi_1 \times \Psi_2$ is a soft directed graph of $\Psi_1^* \times \Psi_2^*$.*

Proof. Let $\Psi_1 = (\Psi_1^*, \gamma_1, \alpha_1, \Re_1) = \{M_1(\varepsilon) : \varepsilon \in \Re_1\}$ be a soft directed graph of $\Psi_1^* = (\varrho_1, \delta_1)$ and $\Psi_2 = (\Psi_2^*, \gamma_2, \alpha_2, \Re_2) = \{M_2(\varepsilon) : \varepsilon \in \Re_2\}$ be a soft directed graph of $\Psi_2^* = (\varrho_2, \delta_2)$. Then the categorical product $\Psi_1 \times \Psi_2$ is defined as,

$$\Psi_1 \times \Psi_2 = \{M_1(\varepsilon_1) \times M_2(\varepsilon_2) : (\varepsilon_1, \varepsilon_2) \in \Re_1 \times \Re_2\}.$$

Here $M_1(\varepsilon_1) \times M_2(\varepsilon_2)$ denotes the categorical product of the directed parts $M_1(\varepsilon_1)$ of Ψ_1 and $M_2(\varepsilon_2)$ of Ψ_2 which is defined as follows: $M_1(\varepsilon_1) \times M_2(\varepsilon_2)$ is a directed graph with node set $\varrho(M_1(\varepsilon_1)\times M_2(\varepsilon_2)) = \gamma_1(\varepsilon_1)\times\gamma_2(\varepsilon_2)$ and directed edge set $\delta(M_1(\varepsilon_1)\times M_2(\varepsilon_2))$, where $((t_1, t_1'), (t_2, t_2'))$ is a directed edge in $M_1(\varepsilon_1) \times M_2(\varepsilon_2)$ if and only if (v_1, v_2) is a directed edge in $M_1(\varepsilon_1)$ and (t_1', t_2') is a directed edge in in $M_2(\varepsilon_2)$.

The categorical product $\Psi_1^* \times \Psi_2^*$ of the two directed graphs Ψ_1^* and Ψ_2^* is a directed graph with node set $\varrho(\Psi_1^* \times \Psi_2^*) = \varrho_1 \times \varrho_2$ and directed edge set $\delta(\Psi_1^* \times \Psi_2^*)$, where $((t_1, t_1'), (t_2, t_2'))$ is a directed edge in $\Psi_1^* \times \Psi_2^*$ if and only if (v_1, v_2) is a directed edge in Ψ_1^* and (t_1', t_2') is a directed edge in in Ψ_2^*.

Let the parameter set be $\Re_{\Psi_1\times\Psi_2} = \Re_1 \times \Re_2$. Define $\gamma_{\Psi_1\times\Psi_2}$ from $\Re_{\Psi_1\times\Psi_2}$ to $\mathcal{P}[\varrho(\Psi_1^* \times \Psi_2^*)]$ by $\gamma_{\Psi_1\times\Psi_2}(\varepsilon_1, \varepsilon_2) = \gamma_1(\varepsilon_1) \times \gamma_2(\varepsilon_2), \forall(\varepsilon_1, \varepsilon_2) \in \Re_1 \times \Re_2$. Then $(\gamma_{\Psi_1\times\Psi_2}, \Re_{\Psi_1\times\Psi_2})$ is a soft set over $\varrho(\Psi_1^* \times \Psi_2^*)$. Also, define $\alpha_{\Psi_1\times\Psi_2}$ from $\Re_{\Psi_1\times\Psi_2}$ to $\mathcal{P}[\delta(\Psi_1^* \times \Psi_2^*)]$ by,

$$\alpha_{\Psi_1\times\Psi_2}(\varepsilon_1, \varepsilon_2) = \{((u, v), (y, z)) \in \delta(\Psi_1^*\times\Psi_2^*) \mid \{(u, v), (y, z)\} \in \gamma_{\Psi_1\times\Psi_2}(\varepsilon_1, \varepsilon_2)\}, \forall(\varepsilon_1, \varepsilon_2) \in \Re_1\times\Re_2.$$

Then $(\alpha_{\Psi_1\times\Psi_2}, \Re_{\Psi_1\times\Psi_2})$ is a soft set over $\delta(\Psi_1^*\times\Psi_2^*)$. If we represent $(\gamma_{\Psi_1\times\Psi_2}(\varepsilon_1, \varepsilon_2), \alpha_{\Psi_1\times\Psi_2}(\varepsilon_1, \varepsilon_2))$ by $M_{\Psi_1\times\Psi_2}(\varepsilon_1, \varepsilon_2)$, then $M_{\Psi_1\times\Psi_2}(\varepsilon_1, \varepsilon_2)$ is a subdirected graph of $\Psi_1^* \times \Psi_2^*, \forall(\varepsilon_1, \varepsilon_2) \in \Re_1 \times \Re_2$, since $\gamma_1(\varepsilon_1) \times \gamma_2(\varepsilon_2) \subseteq \varrho_1 \times \varrho_2$ and any directed edge in $\alpha_{\Psi_1\times\Psi_2}(\varepsilon_1, \varepsilon_2)$ is in $\delta(\Psi_1^* \times \Psi_2^*)$. Then $\Psi_1 \times \Psi_2$ can be represented by the 4-tuple $(\Psi_1^* \times \Psi_2^*, \gamma_{\Psi_1\times\Psi_2}, \alpha_{\Psi_1\times\Psi_2}, \Re_{\Psi_1\times\Psi_2})$ and also by,

$$\{M_{\Psi_1\times\Psi_2}(\varepsilon_1, \varepsilon_2) : (\varepsilon_1, \varepsilon_2) \in \Re_1 \times \Re_2\}$$

and $\Psi_1 \times \Psi_2$ is a soft directed graph of $\Psi_1^* \times \Psi_2^*$. □

Theorem 3.2. *The categorical product $\Psi_1 \times \Psi_2$ contains $\sum_{(\varepsilon_i,\varepsilon_j)\in\Re_1\times\Re_2} |\gamma_1(\varepsilon_i)||\gamma_2(\varepsilon_j)|$ nodes and $\sum_{(\varepsilon_i,\varepsilon_j)\in\Re_1\times\Re_2} |\alpha_1(\varepsilon_i)||\alpha_2(\varepsilon_j)|$ directed edges, if we count the nodes and directed edges based on the number of times they appear in various directed parts of $\Psi_1 \times \Psi_2$.*

Proof. By definition, $\Psi_1 \times \Psi_2 = \{M_1(\varepsilon_1) \times M_2(\varepsilon_2) : (\varepsilon_1, \varepsilon_2) \in \Re_1 \times \Re_2\}$. The parameter set of $\Psi_1 \times \Psi_2$ is $\Re_1 \times \Re_2$. Consider the directed part $M_1(\varepsilon_i) \times M_2(\varepsilon_j)$ of $\Psi_1 \times \Psi_2$ corresponding to the parameter $(\varepsilon_i, \varepsilon_j) \in \Re_1 \times \Re_2$. The node set of $M_1(\varepsilon_i) \times M_2(\varepsilon_j)$ is $\gamma_1(\varepsilon_i) \times \gamma_2(\varepsilon_j)$ which contains $|\gamma_1(\varepsilon_i)||\gamma_2(\varepsilon_j)|$ elements. This is the case for all directed parts of $\Psi_1 \times \Psi_2$. Therefore the total number of nodes in $\Psi_1 \times \Psi_2$ is $\sum_{(\varepsilon_i,\varepsilon_j)\in\Re_1\times\Re_2} |\gamma_1(\varepsilon_i)||\gamma_2(\varepsilon_j)|$.

Also we know, there is a directed edge $((t_q, t_r), (t_s, t_w))$ in $M_1(\varepsilon_i) \times M_2(\varepsilon_j)$ if and only if (t_q, t_s) is a directed edge in $M_1(\varepsilon_i)$ and (t_r, t_w) is a directed edge in $M_2(\varepsilon_j)$. There are $|\alpha_1(\varepsilon_i)|$ directed edges in $M_1(\varepsilon_i)$ and $|\alpha_2(\varepsilon_j)|$ directed edges in $M_2(\varepsilon_j)$. So we can choose a pair of directed edges a_k and a_l such that one is from $M_1(\varepsilon_i)$ and the other is from $M_2(\varepsilon_j)$ in $|\alpha_1(\varepsilon_i)||\alpha_2(\varepsilon_j)|$ different ways. Suppose that a_k is the directed edge (t_q, t_s) in $M_1(\varepsilon_i)$ and a_l is the directed edge (t_r, t_w) in $M_2(\varepsilon_j)$. Then this pair of directed edges gives a directed edge $((t_q, t_r), (t_s, t_w))$ in $M_1(\varepsilon_i) \times M_2(\varepsilon_j)$. Hence $M_1(\varepsilon_i) \times M_2(\varepsilon_j)$ contains totally $|\alpha_1(\varepsilon_i)||\alpha_2(\varepsilon_j)|$ directed edges. This is the case for all directed parts of $\Psi_1 \times \Psi_2$. Therefore total number of directed edges in $\Psi_1 \times \Psi_2$ is

$$\sum_{(\varepsilon_i,\varepsilon_j)\in\Re_1\times\Re_2} |\alpha_1(\varepsilon_i)||\alpha_2(\varepsilon_j)|.$$

□

Example 3.2. *Consider the directed graphs given in Example 3.1. Here we have total number of nodes in $\Psi_1 \times \Psi_2 = 12$ and $\sum_{(\varepsilon_i,\varepsilon_j)\in\Re_1\times\Re_2} |\gamma_1(\varepsilon_i)||\gamma_2(\varepsilon_j)| = (3.2) + (3.2) = 12$. That is, the total number of nodes in $\Psi_1 \times \Psi_2 = \sum_{(\varepsilon_i,\varepsilon_j)\in\Re_1\times\Re_2} |\gamma_1(\varepsilon_i)||\gamma_2(\varepsilon_j)|$.*

Also total number of directed edges in $\Psi_1 \times \Psi_2 = 5$ *and,*

$$\sum_{(\varepsilon_i,\varepsilon_j)\in\Re_1\times\Re_2} |\alpha_1(\varepsilon_i)||\alpha_2(\varepsilon_j)| = (3.1) + (2.1) = 5.$$

That is, total number of directed edges in $\Psi_1 \times \Psi_2 = \sum_{(\varepsilon_i,\varepsilon_j)\in\Re_1\times\Re_2} |\alpha_1(\varepsilon_i)||\alpha_2(\varepsilon_j)|$.

Theorem 3.3. *Let* $\Psi_1^* = (\varrho_1, \delta_1)$ *and* $\Psi_2^* = (\varrho_2, \delta_2)$ *be two directed graphs and* $\Psi_1 = (\Psi_1^*, \gamma_1, \alpha_1, \Re_1)$ *and* $\Psi_2 = (\Psi_2^*, \gamma_2, \alpha_2, \Re_2)$ *be two soft directed graphs of* Ψ_1^* *and* Ψ_2^* *respectively. Then,*

(*i*)

$$\begin{aligned}
&\sum_{(\varepsilon_i,\varepsilon_j)\in\Re_1\times\Re_2} \sum_{(u,v)\in\gamma_{\Psi_1\times\Psi_2}(\varepsilon_i,\varepsilon_j)} ideg(u,v)[M_{\Psi_1\times\Psi_2}(\varepsilon_i,\varepsilon_j)] \\
&= \sum_{(\varepsilon_i,\varepsilon_j)\in\Re_1\times\Re_2} \sum_{(u,v)\in\gamma_{\Psi_1\times\Psi_2}(\varepsilon_i,\varepsilon_j)} odeg(u,v)[M_{\Psi_1\times\Psi_2}(\varepsilon_i,\varepsilon_j)] \\
&= \sum_{(\varepsilon_i,\varepsilon_j)\in\Re_1\times\Re_2} |\alpha_1(\varepsilon_i)||\alpha_2(\varepsilon_j)|.
\end{aligned}$$

(*ii*)

$$\sum_{(\varepsilon_i,\varepsilon_j)\in\Re_1\times\Re_2} \sum_{(u,v)\in\gamma_{\Psi_1\times\Psi_2}(\varepsilon_i,\varepsilon_j)} deg(u,v)[M_{\Psi_1\times\Psi_2}(\varepsilon_i,\varepsilon_j)] = \sum_{(\varepsilon_i,\varepsilon_j)\in\Re_1\times\Re_2} 2|\alpha_1(\varepsilon_i)||\alpha_2(\varepsilon_j)|.$$

Proof.

(i) Consider any directed part $M_{\Psi_1\times\Psi_2}(\varepsilon_i,\varepsilon_j) = (\gamma_{\Psi_1\times\Psi_2}(\varepsilon_i,\varepsilon_j), \alpha_{\Psi_1\times\Psi_2}(\varepsilon_i,\varepsilon_j))$ of $\Psi_1 \times \Psi_2$ which is given by $M_1(\varepsilon_i) \times M_2(\varepsilon_j)$. By Theorem 3.2, we have number of directed edges in $M_1(\varepsilon_i) \times M_2(\varepsilon_j)$ is $|\alpha_1(\varepsilon_i)||\alpha_2(\varepsilon_j)|$. Since the directed part $M_{\Psi_1\times\Psi_2}(\varepsilon_i,\varepsilon_j)$ is a directed graph having $|\alpha_1(\varepsilon_i)||\alpha_2(\varepsilon_j)|$ directed edges, we have

$$\begin{aligned}
\sum_{(u,v)\in\gamma_{\Psi_1\times\Psi_2}(\varepsilon_i,\varepsilon_j)} ideg(u,v)[M_{\Psi_1\times\Psi_2}(\varepsilon_i,\varepsilon_j)] &= \sum_{(u,v)\in\gamma_{\Psi_1\times\Psi_2}(\varepsilon_i,\varepsilon_j)} odeg(u,v)[M_{\Psi_1\times\Psi_2}(\varepsilon_i,\varepsilon_j)] \\
&= |\alpha_1(\varepsilon_i)||\alpha_2(\varepsilon_j)|,
\end{aligned}$$

since each directed edge in $M_{\Psi_1\times\Psi_2}(\varepsilon_i,\varepsilon_j)$ contributes 1 each to the sum,

$$\sum_{(u,v)\in\gamma_{\Psi_1\times\Psi_2}(\varepsilon_i,\varepsilon_j)} ideg(u,v)[M_{\Psi_1\times\Psi_2}(\varepsilon_i,\varepsilon_j)],$$

and to the sum,

$$\sum_{(u,v)\in\gamma_{\Psi_1\times\Psi_2}(\varepsilon_i,\varepsilon_j)} odeg(u,v)[M_{\Psi_1\times\Psi_2}(\varepsilon_i,\varepsilon_j)].$$

This is the case for all the directed parts $M_{\Psi_1\times\Psi_2}(\varepsilon_i,\varepsilon_j)$ of $\Psi_1 \times \Psi_2$. Hence,

$$\begin{aligned}
&\sum_{(\varepsilon_i,\varepsilon_j)\in\Re_1\times\Re_2} \sum_{(u,v)\in\gamma_{\Psi_1\times\Psi_2}(\varepsilon_i,\varepsilon_j)} ideg(u,v)[M_{\Psi_1\times\Psi_2}(\varepsilon_i,\varepsilon_j)] \\
&= \sum_{(\varepsilon_i,\varepsilon_j)\in\Re_1\times\Re_2} \sum_{(u,v)\in\gamma_{\Psi_1\times\Psi_2}(\varepsilon_i,\varepsilon_j)} odeg(u,v)[M_{\Psi_1\times\Psi_2}(\varepsilon_i,\varepsilon_j)] \\
&= \sum_{(\varepsilon_i,\varepsilon_j)\in\Re_1\times\Re_2} |\alpha_1(\varepsilon_i)||\alpha_2(\varepsilon_j)|.
\end{aligned}$$

(ii) Since $deg(u,v)[M_{\Psi_1\times\Psi_2}(\varepsilon_i,\varepsilon_j)] = ideg(u,v)[M_{\Psi_1\times\Psi_2}(\varepsilon_i,\varepsilon_j)] + odeg(u,v)[M_{\Psi_1\times\Psi_2}(\varepsilon_i,\varepsilon_j)]$ and by part (i) of this theorem we have

$$\sum_{(\varepsilon_i,\varepsilon_j)\in\Re_1\times\Re_2}\ \sum_{(u,v)\in\gamma_{\Psi_1\times\Psi_2}(\varepsilon_i,\varepsilon_j)} deg(u,v)[M_{\Psi_1\times\Psi_2}(\varepsilon_i,\varepsilon_j)] = \sum_{(\varepsilon_i,\varepsilon_j)\in\Re_1\times\Re_2} 2|\alpha_1(\varepsilon_i)||\alpha_2(\varepsilon_j)|.$$

□

Example 3.3. *Consider the directed graphs given in Example 3.1. Here we have*

$$\sum_{(\varepsilon_i,\varepsilon_j)\in\Re_1\times\Re_2}\ \sum_{(u,v)\in\gamma_{\Psi_1\times\Psi_2}(\varepsilon_i,\varepsilon_j)} ideg(u,v)[M_{\Psi_1\times\Psi_2}(\varepsilon_i,\varepsilon_j)] = 3+2=5,$$

$$\sum_{(\varepsilon_i,\varepsilon_j)\in\Re_1\times\Re_2}\ \sum_{(u,v)\in\gamma_{\Psi_1\times\Psi_2}(\varepsilon_i,\varepsilon_j)} odeg(u,v)[M_{\Psi_1\times\Psi_2}(\varepsilon_i,\varepsilon_j)] = 3+2=5,$$

$$\sum_{(\varepsilon_i,\varepsilon_j)\in\Re_1\times\Re_2} |\alpha_1(\varepsilon_i)||\alpha_2(\varepsilon_j)| = 3.1+2.1=5.$$

That is,

$$\sum_{(\varepsilon_i,\varepsilon_j)\in\Re_1\times\Re_2}\ \sum_{(u,v)\in\gamma_{\Psi_1\times\Psi_2}(\varepsilon_i,\varepsilon_j)} ideg(u,v)[M_{\Psi_1\times\Psi_2}(\varepsilon_i,\varepsilon_j)]$$
$$= \sum_{(\varepsilon_i,\varepsilon_j)\in\Re_1\times\Re_2}\ \sum_{(u,v)\in\gamma_{\Psi_1\times\Psi_2}(\varepsilon_i,\varepsilon_j)} odeg(u,v)[M_{\Psi_1\times\Psi_2}(\varepsilon_i,\varepsilon_j)] = \sum_{(\varepsilon_i,\varepsilon_j)\in\Re_1\times\Re_2} |\alpha_1(\varepsilon_i)||\alpha_2(\varepsilon_j)|.$$

Also,

$$\sum_{(\varepsilon_i,\varepsilon_j)\in\Re_1\times\Re_2}\ \sum_{(u,v)\in\gamma_{\Psi_1\times\Psi_2}(\varepsilon_i,\varepsilon_j)} deg(u,v)[M_{\Psi_1\times\Psi_2}(\varepsilon_i,\varepsilon_j)] = 6+4=10,$$

$$\sum_{(\varepsilon_i,\varepsilon_j)\in\Re_1\times\Re_2} 2|\alpha_1(\varepsilon_i)||\alpha_2(\varepsilon_j)| = 2.3.1+2.2.1=10.$$

That is,

$$\sum_{(\varepsilon_i,\varepsilon_j)\in\Re_1\times\Re_2}\ \sum_{(u,v)\in\gamma_{\Psi_1\times\Psi_2}(\varepsilon_i,\varepsilon_j)} deg(u,v)[M_{\Psi_1\times\Psi_2}(\varepsilon_i,\varepsilon_j)] = \sum_{(\varepsilon_i,\varepsilon_j)\in\Re_1\times\Re_2} 2|\alpha_1(\varepsilon_i)||\alpha_2(\varepsilon_j)|.$$

4 Restricted Categorical Product of Soft Directed Graphs

In this section, we introduce the concept of the restricted categorical product of soft directed graphs and present several fundamental theorems related to its structure and properties. We start with the definition of the restricted categorical product, $\Psi_1 \otimes \Psi_2$, formed from two soft directed graphs Ψ_1 and Ψ_2 derived from a common directed graph Ψ^*. This product is defined over the intersection of the parameter sets of the two soft directed graphs. We then establish three key theorems: the first confirms that $\Psi_1 \otimes \Psi_2$ is itself a soft directed graph; the second theorem provides a formula for the number of nodes and directed edges in $\Psi_1 \otimes \Psi_2$; and the third theorem offers insights into various degree sums of the nodes in the restricted categorical product. To illustrate these theoretical results, a few examples will be shown.

Definition 4.1. *Let $\Psi^* = (\varrho, \delta)$ be a directed graph and*

$$\Psi_1 = (\Psi^*, \gamma_1, \alpha_1, \Re_1) = \{M_1(\varepsilon) : \varepsilon \in \Re_1\}, \quad \text{and} \quad \Psi_2 = (\Psi^*, \gamma_2, \alpha_2, \Re_2) = \{M_2(\varepsilon) : \varepsilon \in \Re_2\},$$

be two soft directed graphs of Ψ^* such that $\Re_1 \cap \Re_2 \neq \phi$. Then the restricted categorical product of Ψ_1 and Ψ_2, which is represented by $\Psi_1 \otimes \Psi_2$, is defined as $\Psi_1 \otimes \Psi_2 = \{M_1(\varepsilon) \times M_2(\varepsilon) : \varepsilon \in \Re_1 \cap \Re_2\}$. Here $M_1(\varepsilon) \times M_2(\varepsilon)$ denotes the categorical product of the directed parts $M_1(\varepsilon)$ of Ψ_1 and $M_2(\varepsilon)$ of Ψ_2 which is defined as follows: $M_1(\varepsilon) \times M_2(\varepsilon)$ is a directed graph with node set $\varrho(M_1(\varepsilon) \times M_2(\varepsilon)) = \gamma_1(\varepsilon) \times \gamma_2(\varepsilon)$ and directed edge set $\delta(M_1(\varepsilon) \times M_2(\varepsilon))$, where $((t_1, t_1'), (t_2, t_2'))$ is a directed edge in $M_1(\varepsilon) \times M_2(\varepsilon)$ if and only if (t_1, t_2) is a directed edge in $M_1(\varepsilon)$ and (t_1', t_2') is a directed edge in $M_2(\varepsilon)$.

Example 4.1. *Let $\Psi^* = (\varrho, \delta)$ be a directed graph which is shown in Figure 6.*

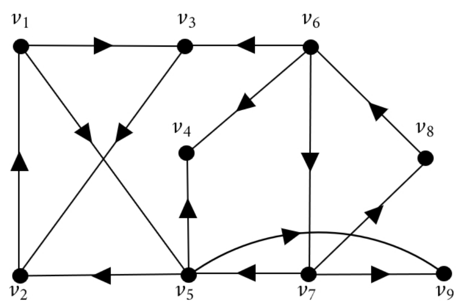

Figure 6: Directed graph $\Psi^* = (\varrho, \delta)$.

Let $\Re_1 = \{v_2, v_6\} \subseteq \varrho$ be a set of parameters. Define,

$$\gamma_1 : \Re_1 \to \mathcal{P}(\varrho) \text{ by } \gamma_1(\varepsilon) = \{u \in \varrho \mid u = \varepsilon \text{ or } u \text{ is adjacent from } \varepsilon \text{ or } u \text{ is adjacent to } \varepsilon\}, \quad \forall \varepsilon \in \Re_1.$$

That is, $\gamma_1(v_2) = \{v_1, v_2, v_3, v_5\}$ and $\gamma_1(v_6) = \{v_3, v_4, v_6, v_7, v_8\}$. Here $(\gamma_1, \Re_1)$ is a soft set over ϱ.

Also, define $\alpha_1 : \Re_1 \to \mathcal{P}(\delta)$ by $\alpha_1(\varepsilon) = \{(u, v) \in \delta \mid \{u, v\} \subseteq \gamma_1(\varepsilon)\}, \forall \varepsilon \in \Re_1$. That is,

$$\alpha_1(v_2) = \{(v_1, v_3), (v_1, v_5), (v_2, v_1), (v_3, v_2), (v_5, v_2)\}, \text{ and}$$
$$\alpha_1(v_6) = \{(v_6, v_3), (v_6, v_4), (v_6, v_7), (v_8, v_6), (v_7, v_8)\}.$$

Here, $(\alpha_1, \Re_1)$ is a soft set over δ. Then, $M_1(v_2) = (\gamma_1(v_2), \alpha_1(v_2))$ and $M_1(v_6) = (\gamma_1(v_6), \alpha_1(v_6))$ are subdirected graphs of Ψ^* as shown in Figure 7. Therefore $\Psi_1 = \{M_1(v_2), M_1(v_6)\}$ is a soft directed graph of Ψ^*.

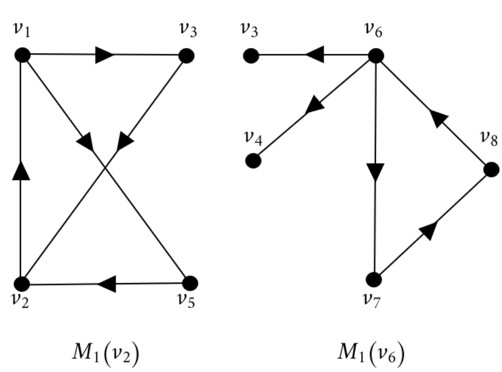

Figure 7: Soft directed graph $\Psi_1 = \{M_1(v_2), M_1(v_6)\}$.

Consider another parameter set $\Re_2 = \{v_1, v_6\} \subseteq \varrho$. *Define,*

$$\gamma_2 : \Re_2 \to \mathcal{P}(\varrho) \text{ by } \gamma_2(\varepsilon) = \{u \in \varrho \mid u = \varepsilon \text{ or } u \text{ is adjacent from } \varepsilon\}, \qquad \forall \varepsilon \in \Re_2.$$

That is, $\gamma_2(v_1) = \{v_1, v_3, v_5\}$ *and* $\gamma_2(v_6) = \{v_3, v_4, v_6, v_7\}$. *Here,* $(\gamma_2, \Re_2)$ *is a soft set over* ϱ.

Also, define $\alpha_2 : \Re_2 \to \mathcal{P}(\delta)$ *by* $\alpha_2(\varepsilon) = \{(u,v) \in \delta \mid \{u,v\} \subseteq \gamma_2(\varepsilon)\}, \forall \varepsilon \in \Re_2$. *That is,* $\alpha_2(v_1) = \{(v_1, v_3), (v_1, v_5)\}$ *and* $\alpha_2(v_6) = \{(v_6, v_3), (v_6, v_4), (v_6, v_7)\}$. *Here,* $(\alpha_2, \Re_2)$ *is a soft set over* δ. *Then,* $M_2(v_1) = (\gamma_2(v_1), \alpha_2(v_1))$ *and* $M_2(v_6) = (\gamma_2(v_6), \alpha_2(v_6))$ *are subdirected graphs of* Ψ^* *as shown in Figure 8. Therefore,* $\Psi_2 = \{M_2(v_1), M_2(v_6)\}$ *is a soft directed graph of* Ψ^*.

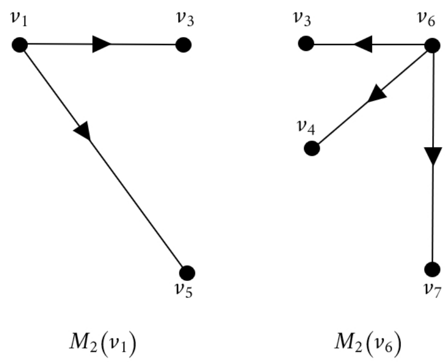

Figure 8: Soft directed graph $\Psi_2 = \{M_2(v_1), M_2(v_6)\}$.

Then, the restricted categorical product $\Psi_1 \otimes \Psi_2 = \{M_1(v_6) \times M_2(v_6)\}$ *is shown in Figure 9.*

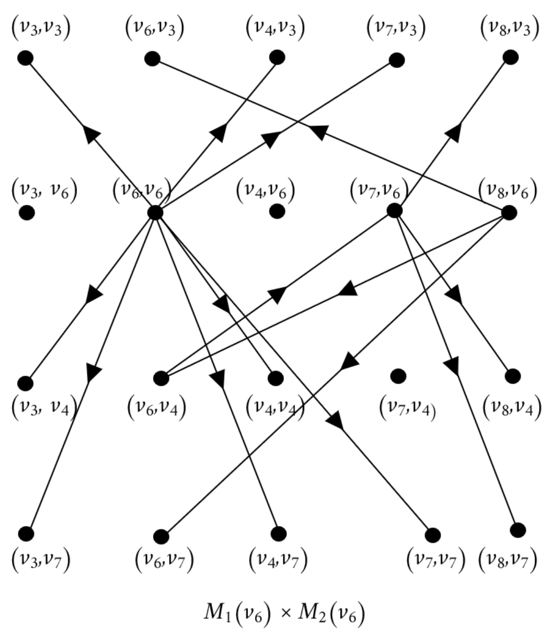

Figure 9: $\Psi_1 \otimes \Psi_2 = \{M_1(v_6) \times M_2(v_6)\}$.

Theorem 4.1. *Let,* $\Psi^* = (\varrho, \delta)$ *be a directed graph and* $\Psi_1 = (\Psi^*, \gamma_1, \alpha_1, \Re_1) = \{M_1(\varepsilon) : \varepsilon \in \Re_1\}$ *and* $\Psi_2 = (\Psi^*, \gamma_2, \alpha_2, \Re_2) = \{M_2(\varepsilon) : \varepsilon \in \Re_2\}$ *be two soft directed graphs of* Ψ^* *such that* $\Re_1 \cap \Re_2 \neq \phi$. *The categorical product* $\Psi_1 \otimes \Psi_2$ *is a soft directed graph of* $\Psi^* \times \Psi^*$.

Proof. Let, $\Psi_1 = (\Psi^*, \gamma_1, \alpha_1, \Re_1) = \{M_1(\varepsilon) : \varepsilon \in \Re_1\}$ and $\Psi_2 = (\Psi^*, \gamma_2, \alpha_2, \Re_2) = \{M_2(\varepsilon) : \varepsilon \in \Re_2\}$ be soft directed graphs of $\Psi^* = (\varrho, \delta)$ such that $\Re_1 \cap \Re_2 \neq \phi$. Then, the restricted categorical product $\Psi_1 \otimes \Psi_2$ is defined as $\Psi_1 \otimes \Psi_2 = \{M_1(\varepsilon) \times M_2(\varepsilon) : \varepsilon \in \Re_1 \cap \Re_2\}$. Here, $M_1(\varepsilon) \times M_2(\varepsilon)$ denotes the categorical product of the directed parts $M_1(\varepsilon)$ of Ψ_1 and $M_2(\varepsilon)$ of Ψ_2 which is defined as follows: $M_1(\varepsilon) \times M_2(\varepsilon)$ is a directed graph with node set $\varrho(M_1(\varepsilon) \times M_2(\varepsilon)) = \gamma_1(\varepsilon) \times \gamma_2(\varepsilon)$ and directed edge set $\delta(M_1(\varepsilon) \times M_2(\varepsilon))$, where $((t_1, t_1'), (t_2, t_2'))$ is a directed edge in $M_1(\varepsilon) \times M_2(\varepsilon)$ if and only if (t_1, t_2) is a directed edge in $M_1(\varepsilon)$ and (t_1', t_2') is a directed edge in $M_2(\varepsilon)$.

The categorical product $\Psi^* \times \Psi^*$ is a directed graph with node set $\varrho(\Psi^* \times \Psi^*) = \varrho \times \varrho$ and directed edge set $\delta(\Psi^* \times \Psi^*)$, where $((t_1, t_1'), (t_2, t_2'))$ is a directed edge in $\Psi^* \times \Psi^*$ if and only if (v_1, v_2) as well as (t_1', t_2') are directed edges in Ψ^*. Let the parameter set be $\Re_{\Psi_1 \otimes \Psi_2} = \Re_1 \cap \Re_2$. Define $\gamma_{\Psi_1 \otimes \Psi_2}$ from $\Re_{\Psi_1 \otimes \Psi_2}$ to $\mathcal{P}[\varrho(\Psi^* \times \Psi^*)]$ by,

$$\gamma_{\Psi_1 \otimes \Psi_2}(\varepsilon) = \gamma_1(\varepsilon) \times \gamma_2(\varepsilon), \forall \varepsilon \in \Re_1 \cap \Re_2.$$

Then, $(\gamma_{\Psi_1 \otimes \Psi_2}, \Re_{\Psi_1 \otimes \Psi_2})$ is a soft set over $\varrho(\Psi^* \times \Psi^*)$. Also, define $\alpha_{\Psi_1 \otimes \Psi_2}$ from $\Re_{\Psi_1 \otimes \Psi_2}$ to $\mathcal{P}[\delta(\Psi^* \times \Psi^*)]$ by,

$$\alpha_{\Psi_1 \otimes \Psi_2}(\varepsilon) = \{((u,v),(y,z)) \in \delta(\Psi^* \times \Psi^*) \mid \{(u,v),(y,z)\} \in \gamma_{\Psi_1 \otimes \Psi_2}(\varepsilon)\}, \qquad \forall \varepsilon \in \Re_1 \cap \Re_2.$$

Then, $(\alpha_{\Psi_1 \otimes \Psi_2}, \Re_{\Psi_1 \otimes \Psi_2})$ is a soft set over $\delta(\Psi^* \times \Psi^*)$. If we represent $(\gamma_{\Psi_1 \otimes \Psi_2}(\varepsilon), \alpha_{\Psi_1 \otimes \Psi_2}(\varepsilon))$ by $M_{\Psi_1 \otimes \Psi_2}(\varepsilon)$, then $M_{\Psi_1 \otimes \Psi_2}(\varepsilon)$ is a subdirected graph of $\Psi^* \times \Psi^*, \forall \varepsilon \in \Re_1 \cap \Re_2$, since,

$$\gamma_1(\varepsilon) \times \gamma_2(\varepsilon) \subseteq \varrho \times \varrho,$$

and any directed edge in $\alpha_{\Psi_1 \otimes \Psi_2}(\varepsilon)$ is also a directed edge in $\delta(\Psi^* \times \Psi^*)$. Then $\Psi_1 \otimes \Psi_2$ can be represented by the 4-tuple $(\Psi^* \times \Psi^*, \gamma_{\Psi_1 \otimes \Psi_2}, \alpha_{\Psi_1 \otimes \Psi_2}, \Re_{\Psi_1 \otimes \Psi_2})$ and also by $\{M_{\Psi_1 \otimes \Psi_2}(\varepsilon) : \varepsilon \in \Re_1 \cap \Re_2\}$ and $\Psi_1 \otimes \Psi_2$ is a soft directed graph of $\Psi^* \times \Psi^*$. □

Theorem 4.2. *The restricted categorical product $\Psi_1 \otimes \Psi_2$ contains $\sum_{\varepsilon \in \Re_1 \cap \Re_2}(|\gamma_1(\varepsilon)||\gamma_2(\varepsilon)|)$ nodes and $\sum_{\varepsilon \in \Re_1 \cap \Re_2}(|\alpha_1(\varepsilon)||\alpha_2(\varepsilon)|)$ directed edges.*

Proof. By definition, $\Psi_1 \otimes \Psi_2 = \{M_1(\varepsilon) \times M_2(\varepsilon) : \varepsilon \in \Re_1 \cap \Re_2\}$. The parameter set of $\Psi_1 \otimes \Psi_2$ is $\Re_1 \cap \Re_2$. Consider the directed part $M_1(\varepsilon) \times M_2(\varepsilon)$ of $\Psi_1 \otimes \Psi_2$ corresponding to the parameter $\varepsilon \in \Re_1 \cap \Re_2$. The node set of $M_1(\varepsilon) \times M_2(\varepsilon)$ is $\gamma_1(\varepsilon) \times \gamma_2(\varepsilon)$ which contains $|\gamma_1(\varepsilon)||\gamma_2(\varepsilon)|$ elements. This is the case for all directed parts of $\Psi_1 \otimes \Psi_2$. Therefore total number of nodes in $\Psi_1 \otimes \Psi_2$ is $\sum_{\varepsilon \in \Re_1 \cap \Re_2} |\gamma_1(\varepsilon)||\gamma_2(\varepsilon)|$.

Also we know, $((t_q, t_r), (t_s, t_w))$ is a directed edge in $M_1(\varepsilon) \times M_2(\varepsilon)$ if and only if (t_q, t_s) is a directed edge in $M_1(\varepsilon)$ and (t_r, t_w) is a directed edge in $M_2(\varepsilon)$. There are $|\alpha_1(\varepsilon)|$ directed edges in $M_1(\varepsilon)$ and $|\alpha_2(\varepsilon)|$ directed edges in $M_2(\varepsilon)$. So we can choose a pair of directed edges a_k and a_l such that one is from $M_1(\varepsilon)$ and the other is from $M_2(\varepsilon)$ in $|\alpha_1(\varepsilon)||\alpha_2(\varepsilon)|$ different ways. Suppose that a_k is the directed edge (t_q, t_s) in $M_1(\varepsilon)$ and a_l is the directed edge (t_r, t_w) in $M_2(\varepsilon)$. Then this pair of directed edges gives a directed edge $((t_q, t_r), (t_s, t_w))$ in $M_1(\varepsilon) \times M_2(\varepsilon)$. Hence $M_1(\varepsilon) \times M_2(\varepsilon)$ contains totally $|\alpha_1(\varepsilon)||\alpha_2(\varepsilon)|$ directed edges. This is the case for all directed parts of $\Psi_1 \otimes \Psi_2$. Therefore total number of directed edges in $\Psi_1 \otimes \Psi_2$ is $\sum_{\varepsilon \in \Re_1 \cap \Re_2} |\alpha_1(\varepsilon)||\alpha_2(\varepsilon)|$. □

Example 4.2. *Consider the directed graphs given in Example 4.1. Here we have total number of nodes in $\Psi_1 \otimes \Psi_2 = 20$ and $\sum_{\varepsilon \in \Re_1 \cap \Re_2} |\gamma_1(\varepsilon)||\gamma_2(\varepsilon)| = (5.4) = 20$. That is, the total number of nodes in $\Psi_1 \otimes \Psi_2 = \sum_{\varepsilon \in \Re_1 \cap \Re_2} |\gamma_1(\varepsilon)||\gamma_2(\varepsilon)|$.*

Also, total number of directed edges in $\Psi_1 \otimes \Psi_2 = 15$ and $\sum_{\varepsilon \in \Re_1 \cap \Re_2} |\alpha_1(\varepsilon)||\alpha_2(\varepsilon)| = (5.3) = 15$. That is, total number of directed edges in $\Psi_1 \otimes \Psi_2 = \sum_{\varepsilon \in \Re_1 \cap \Re_2} |\alpha_1(\varepsilon)||\alpha_2(\varepsilon)|$.

Theorem 4.3. *Let $\Psi^* = (\varrho, \delta)$ be a directed graph and $\Psi_1 = (\Psi^*, \gamma_1, \alpha_1, \Re_1)$ and $\Psi_2 = (\Psi^*, \gamma_2, \alpha_2, \Re_2)$ be two soft directed graphs of Ψ^* such that $\Re_1 \cap \Re_2 \neq \phi$. Then,*

(*i*)

$$\sum_{\varepsilon \in \Re_1 \cap \Re_2} \sum_{(u,v) \in \gamma_{\Psi_1 \otimes \Psi_2}(\varepsilon)} ideg(u,v)[M_{\Psi_1 \otimes \Psi_2}(\varepsilon)] = \sum_{\varepsilon \in \Re_1 \cap \Re_2} \sum_{(u,v) \in \gamma_{\Psi_1 \otimes \Psi_2}(\varepsilon)} odeg(u,v)[M_{\Psi_1 \otimes \Psi_2}(\varepsilon)] = \sum_{\varepsilon \in \Re_1 \cap \Re_2} |\alpha_1(\varepsilon)||\alpha_2(\varepsilon)|.$$

(*ii*)

$$\sum_{\varepsilon \in \Re_1 \cap \Re_2} \sum_{(u,v) \in \gamma_{\Psi_1 \otimes \Psi_2}(\varepsilon)} deg(u,v)[M_{\Psi_1 \otimes \Psi_2}(\varepsilon)] = \sum_{\varepsilon \in \Re_1 \cap \Re_2} 2|\alpha_1(\varepsilon)||\alpha_2(\varepsilon)|.$$

Proof.

(i) Consider any directed part $M_{\Psi_1 \otimes \Psi_2}(\varepsilon) = (\gamma_{\Psi_1 \times \Psi_2}(\varepsilon), \alpha_{\Psi_1 \times \Psi_2}(\varepsilon))$ of $\Psi_1 \otimes \Psi_2$ which is given by $M_1(\varepsilon) \times M_2(\varepsilon)$. By Theorem 4.2, we have number of directed edges in $M_1(\varepsilon) \times M_2(\varepsilon)$ is $|\alpha_1(\varepsilon)||\alpha_2(\varepsilon)|$. Since the directed part $M_{\Psi_1 \otimes \Psi_2}(\varepsilon)$ is a directed graph having $|\alpha_1(\varepsilon)||\alpha_2(\varepsilon)|$ directed edges, we have

$$\sum_{(u,v) \in \gamma_{\Psi_1 \otimes \Psi_2}(\varepsilon)} ideg(u,v)[M_{\Psi_1 \otimes \Psi_2}(\varepsilon)] = \sum_{(u,v) \in \gamma_{\Psi_1 \otimes \Psi_2}(\varepsilon)} odeg(u,v)[M_{\Psi_1 \otimes \Psi_2}(\varepsilon)] = |\alpha_1(\varepsilon)||\alpha_2(\varepsilon)|,$$

since each directed edge in $M_{\Psi_1 \otimes \Psi_2}(\varepsilon)$ contributes 1 each to the sums

$$\sum_{(u,v) \in \gamma_{\Psi_1 \otimes \Psi_2}(\varepsilon)} ideg(u,v)[M_{\Psi_1 \otimes \Psi_2}(\varepsilon)], \quad \text{and} \quad \sum_{(u,v) \in \gamma_{\Psi_1 \otimes \Psi_2}(\varepsilon)} odeg(u,v)[M_{\Psi_1 \otimes \Psi_2}(\varepsilon)].$$

This is the case for all the directed parts $M_{\Psi_1 \otimes \Psi_2}(\varepsilon)$ of $\Psi_1 \otimes \Psi_2$. Hence,

$$\sum_{\varepsilon \in \Re_1 \cap \Re_2} \sum_{(u,v) \in \gamma_{\Psi_1 \otimes \Psi_2}(\varepsilon)} ideg(u,v)[M_{\Psi_1 \otimes \Psi_2}(\varepsilon)] = \sum_{\varepsilon \in \Re_1 \cap \Re_2} \sum_{(u,v) \in \gamma_{\Psi_1 \otimes \Psi_2}(\varepsilon)} odeg(u,v)[M_{\Psi_1 \otimes \Psi_2}(\varepsilon)] = \sum_{\varepsilon \in \Re_1 \cap \Re_2} |\alpha_1(\varepsilon)||\alpha_2(\varepsilon)|.$$

(ii) Since $deg(u,v)[M_{\Psi_1 \otimes \Psi_2}(\varepsilon)] = ideg(u,v)[M_{\Psi_1 \otimes \Psi_2}(\varepsilon)] + odeg(u,v)[M_{\Psi_1 \otimes \Psi_2}(\varepsilon)]$ and by part (i) of this theorem we have

$$\sum_{\varepsilon \in \Re_1 \cap \Re_2} \sum_{(u,v) \in \gamma_{\Psi_1 \otimes \Psi_2}(\varepsilon)} deg(u,v)[M_{\Psi_1 \otimes \Psi_2}(\varepsilon)] = \sum_{\varepsilon \in \Re_1 \cap \Re_2} 2|\alpha_1(\varepsilon)||\alpha_2(\varepsilon)|.$$

□

Example 4.3. *Consider the directed graphs given in Example 4.1. Here, we have*

$$\sum_{\varepsilon \in \Re_1 \cap \Re_2} \sum_{(u,v) \in \gamma_{\Psi_1 \otimes \Psi_2}(\varepsilon)} ideg(u,v)[M_{\Psi_1 \otimes \Psi_2}(\varepsilon)] = 15,$$

$$\sum_{\varepsilon \in \Re_1 \cap \Re_2} \sum_{(u,v) \in \gamma_{\Psi_1 \otimes \Psi_2}(\varepsilon)} odeg(u,v)[M_{\Psi_1 \otimes \Psi_2}(\varepsilon)] = 15,$$

$$\sum_{\varepsilon \in \Re_1 \cap \Re_2} |\alpha_1(\varepsilon)||\alpha_2(\varepsilon)| = 5.3 = 15.$$

That is,

$$\sum_{\varepsilon\in\Re_1\cap\Re_2}\sum_{(u,v)\in\gamma_{\Psi_1\otimes\Psi_2}(\varepsilon)} ideg(u,v)[M_{\Psi_1\otimes\Psi_2}(\varepsilon)] = \sum_{\varepsilon\in\Re_1\cap\Re_2}\sum_{(u,v)\in\gamma_{\Psi_1\otimes\Psi_2}(\varepsilon)} odeg(u,v)[M_{\Psi_1\otimes\Psi_2}(\varepsilon)] = \sum_{\varepsilon\in\Re_1\cap\Re_2} |\alpha_1(\varepsilon)||\alpha_2(\varepsilon)|.$$

Also,

$$\sum_{\varepsilon\in\Re_1\cap\Re_2}\sum_{(u,v)\in\gamma_{\Psi_1\otimes\Psi_2}(\varepsilon)} deg(u,v)[M_{\Psi_1\otimes\Psi_2}(\varepsilon)] = 30,$$

$$\sum_{\varepsilon\in\Re_1\cap\Re_2} 2|\alpha_1(\varepsilon)||\alpha_2(\varepsilon)| = 2.5.3 = 30.$$

That is,

$$\sum_{\varepsilon\in\Re_1\cap\Re_2}\sum_{(u,v)\in\gamma_{\Psi_1\otimes\Psi_2}(\varepsilon)} deg(u,v)[M_{\Psi_1\otimes\Psi_2}(\varepsilon)] = \sum_{\varepsilon\in\Re_1\cap\Re_2} 2|\alpha_1(\varepsilon)||\alpha_2(\varepsilon)|.$$

5 Conclusion

The study of graphs has revolutionized our understanding and application of relationships in complex systems, from social networking to transportation logistics. The introduction of soft set theory has enhanced our ability to manage uncertainty in these relationships, leading to innovative decision-making solutions. Extending traditional graph models to incorporate uncertainty, researchers have introduced soft directed graphs, allowing for the analysis of uncertain relationships between entities. This study explores both the categorical product and the restricted categorical product of soft directed graphs, defining these products and demonstrating that they form soft directed graphs. We established key theorems detailing the node and edge counts in both products. Additionally, we provided formulas for calculating the directed part in-degree, out-degree, and degree sums of nodes in both products, showing how these degrees are influenced by the structure of the original graphs. By illustrating these theoretical results with practical examples, we demonstrated the utility and implications of our work. These findings significantly advance the theoretical foundation of soft directed graphs.

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